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General stability conditions for zonal flows in a one-layer model on the β-plane or the sphere

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Sufficient stability conditions are derived for a zonal flow on the β-plane or the sphere. Two conditions guarantee both *shear stability* (to perturbations with vanishing zonal average) and *inertial stability* (to longitude-independent perturbations). These conditions are not restricted to normal-mode disturbances, and are derived without making use of the quasi-geostrophic approximation. The main limitation of the model is to have only one layer.

On the β-plane, the conditions are: (i) that the product of the meridional gradient of potential vorticity and the difference between an arbitrary constant and the zonal velocity be everywhere non-negative; *and* (ii) that the absolute value of this difference be nowhere larger than the local phase speed of long gravity waves. Inertial stability is independently assured if the Cariolis parameter and the potential vorticity are everywhere of the same sign (this well-known condition can be easily violated near the equator, but the flow may nonetheless be stable).

If the meridional gradient of potential vorticity has everywhere the same sign, then conditions (i) and (ii) can be shown to be consequences of the conservation of a total pseudo-energy E_0 and pseudomomentum P_0, defined so that their lowest-order contribution is *quadratic* in the deviation from the fundamental state (even in the case that the perturbation is longitude-independent). Thus, if there exists a value of α such that the integral of $E_0 - \alpha P_0$ is positive-definite, then the flow is stable. In this case, the stability conditions are valid for small, rather than infinitesimal, perturbations.

The parameters of stable flows, as guaranteed by these conditions, are investigated for the family of Gaussian jets centred at the equator; both the cases of an unbounded ocean and a semi-infinite ocean, poleward from a zonal wall, are considered. Easterlies with the width of a Kelvin wave and westerlies with that width or wider may be unstable, even though the gradient of potential vorticity is positive for any strength of the jet.

1. Introduction

It is now a little over a century since Rayleigh (1880) showed that a parallel flow $U(y)$ without inflexion points is stable.† Kuo (1949) generalized this result to the non-divergent case with rotation: a sufficient condition of stability is that the gradient of absolute vorticity, $\beta - \partial_{yy} U$, does not change sign. (The notation is standard and is detailed in §1.1.) Lipps (1963) further extended the theorem by including divergence effects: stability is assured when the gradient of potential vorticity

$$(\partial_y Q)_{\mathrm{qg}} = [\beta - \partial_{yy} U + f_0^2 c^{-2} U]/\bar{h} \tag{1}$$

† Fjørtoft (1950) strengthened Rayleigh's criteria showing that if, say, $\partial_{yy} U = 0$ at $y = y^*$, but $[U(y) - U(y^*)]\, \partial_{yy} U \geq 0$ everywhere, then the flow is stable. See Drazin & Howard (1966) for a review of stability conditions, including extensions of the Rayleigh–Kuo criteria to non-parallel flows.

does not change sign. Unfortunately, since Lipps needed the quasi-geostrophic approximation in order to derive (1), it becomes invalid near the equator and/or when divergence effects are too large, as does its generalization to the fully three-dimensional flow.

The most important difference between equatorial and extra-equatorial dynamics is due to the divergence of the horizontal flow: non-divergent motion is governed, at any latitude, by the equation of absolute vorticity conservation, in which f_0 plays no role. (The variation of β with latitude is certainly of no dynamical importance.)

Philander (1976, 1978) has investigated the relative importance of the β and divergence effects on the stability of typical currents in the tropical oceans, mainly via the numerical solution of the eigenvalue problem. The author finds that both β and divergence stabilize eastward jets and may destabilize westward currents. Moreover these effects are enhanced as the equator is approached and ageostrophic phenomena become important.

Hughes (1979, 1981) presented in instability mechanism for the equatorial undercurrent based on the existence of two different flows for a given potential-vorticity–stream-function relationship. The author shows that the narrower of the two currents is unstable and supercritical to Kelvin-like perturbations, exactly the opposite being true for the wider solution.

Semtner & Holland (1980) have also found a barotropically unstable equatorial undercurrent in a numerical model of tropical circulation. The type of instability was identified via an energy-flow calculation.

None of these authors quotes a stability condition appropriate to low latitudes. Miyata (1981) found one, which is essentially a generalization, including the effect of rotation, of Blumen's (1970) work on the instability of two-dimensional compressible flow (a problem that can be related to the two-dimensional incompressible but divergent one). Blumen's and Miyata's conditions are particular cases of the more general one that is reported here. The generality of the condition in this paper is crucial for the determination of the stability of equatorial currents (e.g. Miyata's result does not guarantee the stability of an eastward jet, no matter how small its amplitude may be.)

In addition to the above mentioned *shear instability* (i.e. to perturbations with vanishing zonal average) a geophysical fluid may be subject to *intertial instability* (to longitude-independent perturbations), as predicted by Rayleigh (1916). The necessary condition for inertial instability on the β-plane (viz that the potential vorticity and Coriolis parameter have opposite signs in some region) is the same at mid-latitude or in the equatorial zone. However, this type of instability is more likely to occur in the latter case, owing to the smallness of the Coriolis parameter; a Rossby number of order unity is required for this process to take place. Apparently, this phenomenon has not gained the attention of oceanographers in the study of the tropics (e.g. this author is not aware of any calculation showing whether the South Equatorial Current is inertially stable or not). Some recent contributions to the subject in the atmospheric-sciences literature are those of Emanuel (1979), Busse & Chen (1981*a*, *b*) and Dunkerton (1981).

The rest of this section is devoted to the presentation of the model equations and an *a priori* scale analysis, included with the purpose of stressing the differences between the equatorial and mid-latitude cases. The stability conditions in the β-plane, deduced in §2, are used in §3 to study the stability of a Gaussian jet, are related to the energy–pseudomomentum conservation in §4 and, finally, are generalized to the case of a sphere in §5. A general discussion is given in §6.

This work is meant as a step in the investigation of the stability of equatorial currents. Owing to the extreme simplicity of the *vertical* structure of the model, the temptation to relate its results to the 'real world' will be resisted. The restriction to only one layer is a strong limitation because it requires the perturbation to have the same vertical structure as the fundamental flow. Hence a comparison with experimental data will be made once the model is extended to the case of three-dimensional flow.

1.1. *The model*

We work with the model of the simplest vertical and horizontal structure, viz one-layer in the β-plane. The free and inviscid evolution of the system is governed by

$$D_t u - fv + g_r \partial_x h = 0, \qquad (2a),$$

$$D_t v + fu + g_r \partial_y h = 0, \qquad (2b),$$

$$D_t h + h(\partial_x u + \partial_y v) = 0, \qquad (2c)$$

where $D_t = \partial_t + u\,\partial_x + v\,\partial_y$. Here (u, v) and (x, y) are the (eastward, northward) velocities and coordinates, h is the layer depth, g_r is the reduced gravity (or the gravity), and f is the Coriolis parameter ($f = f_0 + \beta y$, where f_0 and β are constants).

The reduced gravity and mean depth $\bar{h}$ are used to define the 'separation constant' $c = (g_r \bar{h})^{\frac{1}{2}}$, which is, in general, a more useful parameter than g_r and $\bar{h}$. Particular solutions of the linearized version of (2) are the zonally propagating Kelvin waves, for which $v = 0$, u is in geostrophic balance and has the structure $u \propto \exp(-f^2/2\beta c)$, and the eastward phase speed is equal to c. (If $f_0 \neq 0$, this solution requires a wall at $y = 0$, with the ocean poleward from the boundary.) The deformation radius is defined as the e-folding length of such a Kelvin wave, namely

$$R = \frac{2c}{[|f_0| + (f_0^2 + 2\beta c)^{\frac{1}{2}}]}. \qquad (3)$$

The value of R provides a lengthscale, while $\frac{1}{2}\beta R^2$ is a speed scale for rotational (Rossby) perturbations. In the equatorial β-plane, $f_0 = 0$,

$$R \equiv R_0 = (2c/\beta)^{\frac{1}{2}}, \qquad (4)$$

and $\frac{1}{2}\beta R^2$ coincides with the speed scale c for inertia–gravity perturbations. In the mid-latitude β-plane, on the other hand, $R \sim c/|f_0|$ (this requires $f_0^2 \gg \frac{1}{2}\beta c$ and implies $R \ll 2|f_0|/\beta$, $\frac{1}{2}\beta R^2 \ll c$). Figures 1 and 2 show the dependence of R upon the latitude (for different values of c) and of R_0 upon c.

The zonal current $U(y)$, whose stability is investigated, must be in geostrophic balance in order to be an exact solution of (2), i.e.

$$fU + g_r \partial_y H = 0, \qquad (5)$$

where $H(y)$ is the layer depth in the fundamental state. The potential vorticity of this flow is given by

$$Q = \frac{f - \partial_y U}{H}, \qquad (6)$$

and its meridional gradient is then found to be

$$\partial_y Q = H^{-1}\left[\beta - \partial_{yy} U + \frac{fQ}{g_r} U\right]. \qquad (7)$$

Note that the expressions between square brackets on the right-hand sides of (1) and (7) differ by $O(U^2)$.

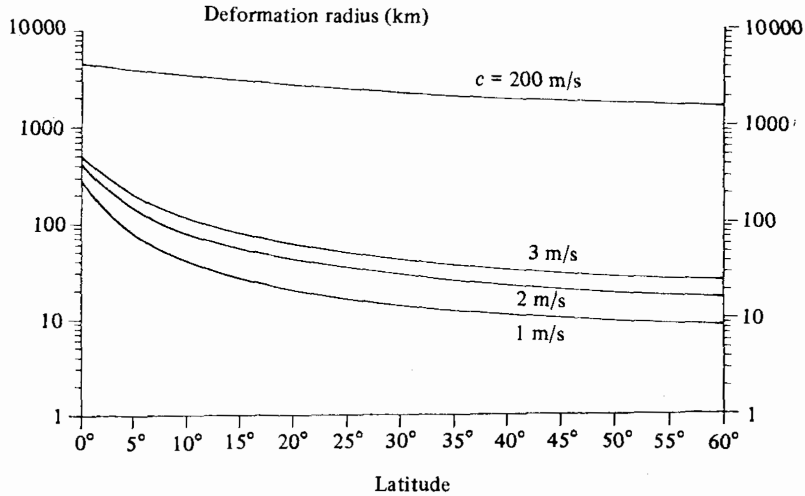

FIGURE 1. Deformation radius for β-plane dynamics at different latitudes. The values of the separation constant c are typical oceanic values for the external and the first three internal modes.

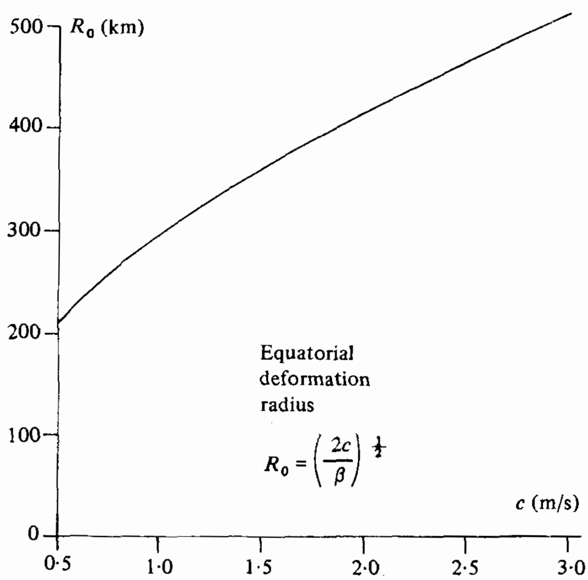

FIGURE 2. Deformation radius for the equatorial β-plane, as a function of the separation constant c.

Let us estimate the amplitude of a jet strong enough to change the sign of the meridional gradient of potential vorticity, for both (1) and (7). We assume $|\partial_{yy} U| \sim |U|/L^2$, where L is the lengthscale for the variations of $U(y)$, and perform a similar scaling in (5), neglecting factors of 2. If $fQ \approx f_0^2/\bar{h}$ in the region of interest, then (1) and (7) are essentially the same, and the amplitude predicted by the quasi-geostrophic approximation is correct. In order for this to happen, it is sufficient that the following three inequalities hold:

$$|f_0| \gg \beta L, \quad |f_0| \gg |\partial_y U|, \quad \bar{h} \gg |H - \bar{h}|. \tag{8a, b, c}$$

In the core of the jet we have $\partial_{yy} U \sim -U/L^2$. At *mid-latitudes*, a flow strong enough to change the sign of the gradient of potential vorticity must have an amplitude, according to (1), of order

$$U \sim -\frac{\beta L^2}{1+(L/R)^2}, \tag{9}$$

where $R \approx c/|f_0|$. Conditions ($8b$, c) are then just consequences of ($8a$), since

$$|\partial_y U| \sim \frac{\beta L}{1+(L/R)^2} \leqslant \beta L, \tag{10}$$

and, using (5),

$$|H-\bar{h}| \sim \frac{|f_0 LU|}{g_r} \sim \frac{\bar{h}|\beta L/f_0|}{1+(R/L)^2} \leqslant \bar{h}\left|\frac{\beta L}{f_0}\right|. \tag{11}$$

Therefore, if ($8a$) is satisfied, the quasi-geostrophic approximation gives the right amplitude in this case.

In the *equatorial region*, on the other hand, a flow strong enough to change the sign of $\partial_y Q$ in the *core* of the jet must have an amplitude of order

$$U \sim -\frac{\beta L^2}{1+(L/R_0)^4}, \tag{12}$$

where we have assumed $fQ \sim (\beta L)^2/\bar{h}$ in (7). (If $fQ \sim 0$, then (12) should be replaced by $U \sim -\beta L^2$.) Moreover, the geostrophic balance (5) implies

$$|H-\bar{h}| \sim \frac{\beta L^2|U|}{g_r} \sim \frac{\bar{h}}{1+(R_0/L)^4}. \tag{13}$$

Consequently, for a current with the width of a Kelvin wave, or wider, ($8c$) is not satisfied, i.e. ageostrophic divergence effects must be important, and the scaling $fQ \sim (\beta L)^2/\bar{h}$ might be incorrect, owing to the large variations of $H(y)$. (Note that $\beta L/|\partial_y U| \sim 1+(L/R_0)^4$ and thus ($8b$), with $|f_0| \sim \beta L$, is satisfied for currents wider than a Kelvin wave, even though ($8a$, c) are violated.)

In the edge of the jet, we have $\partial_{yy} U \sim U/L^2$; divergence and β-effects have the same sign in the gradient of potential vorticity (assuming that $fQ \geqslant 0$). As a consequence, the factors $[1+\dots]$ should be replaced by $[-1+\dots]$ in (9)–(13): the scaling predicts extremely large amplitudes if $L \approx R$ (the quasi-geostrophic approximation is invalid for such a strong flow). Note that (9) and (12) predict a *westward* flow for instability at the core of a jet of any width or (replacing $1+\dots$ by $-1+\dots$) at the edge of a wide jet ($L > R$), and *eastward* flow at the edge of a narrow jet ($L < R$).

In summary, extrema of potential vorticity may be expected at the *core* of easterlies of any width, and at the *edge* of narrow westerlies or wide easterlies (the widthscale is the local deformation radius). The quasi-geostrophic approximation gives the right amplitude for narrow jets ($L \ll R$) at any latitude (for which divergence is unimportant, and Kuo's criterion applies) or wide easterlies ($L \gg R$) narrower than the 'distance to the equator' ($L \ll |f_0|/\beta$).

There are flows without extrema of potential vorticity, independently of the strength of the current. For instance, wide westerlies or narrow easterlies without a 'core' (e.g. a current profile without inflexion points, trapped poleward from a zonal wall). It is interesting to see if such flows might, nevertheless, become unstable. Stability studies based on the primitive equations, rather than the quasi-geostrophic approximation, are needed to answer this question, as well as to investigate the dynamics of tropical currents.

2. Stability conditions in the β-plane

The problem we are concerned with is the linear stability of a system with velocity given by $U(y)$ in the zonal direction, and depth $H(y)$. More precisely, writing

$$u = U(y) + u'(x,y,t) + O(\epsilon^2), \tag{14a}$$

$$v = v'(x,y,t) + O(\epsilon^2), \tag{14b}$$

$$h = H(y) + h'(x,y,t) + O(\epsilon^2), \tag{14c}$$

where $(U,H) = Q(\epsilon^0)$, $(u',v',h') = O(\epsilon)$ and $\epsilon \ll 1$, we search for the conditions that (U,H) must satisfy in order for (u',v',h') to be bounded in time. The evolution of the perturbation fields is controlled by

$$(\partial_t + U\partial_x)u' - QHv' + g_r\,\partial_x H' = 0, \tag{15a}$$

$$(\partial_t + U\partial_x)v' + fu' + g_r\partial_y H' = 0, \tag{15b}$$

$$(\partial_t + U\partial_x)h' + \partial_x(u'H) + \partial_y(v'H) = 0. \tag{15c}$$

Two cases are allowed for the zonal structure of the perturbation, namely

$$\langle u'\rangle = \langle v'\rangle = \langle h'\rangle = 0 \quad \text{(shear instability)}, \tag{16a}$$

$$\partial_x u' = \partial_x v' = \partial_x h' = 0 \quad \text{(inertial stability)}, \tag{16b}$$

where $\langle\ \ \rangle$ denotes zonal average. (The notation, considering 'shear' and 'inertial' instabilities as different things, corresponds to that used by Busse & Chen (1981*a*, *b*). Unfortunately, in the literature both terms are sometimes used with the same meaning.)

The nonlinear system under consideration satisfies many conservation laws (Ripa 1982), particularly those of potential vorticity, total energy and total zonal momentum, which take the form

$$D_t q = 0, \quad \int dy\,\langle E\rangle = \text{const}, \quad \int dy\,\langle M\rangle = \text{const}, \tag{17a, b, c}$$

where

$$q = (f + \partial_x v - \partial_y u)/h, \tag{18a}$$

$$E = \tfrac{1}{2}h(u^2+v^2) + \tfrac{1}{2}g_r(h-\bar{h})^2, \tag{18b}$$

$$M = h(u - \tfrac{1}{2}f^2/\beta) \tag{18c}$$

are the potential vorticity and the energy and zonal momentum densities. (It is easier to see the physical meaning of the term $-\tfrac{1}{2}f^2/\beta$ in M using spherical coordinates: it represents the variation of the earth's zonal velocity with latitude; see (77) below. The equations 'on the β-plane' resemble more those of a charged particle on a magnetic field than a rotating frame (because curvature effects are retained only in f): the momentum of such a particle is $m\mathbf{v} + \mathbf{A}$, and not just $m\mathbf{v}$, where $\mathbf{A}$ is the vector potential of the magnetic field (times the charge of the particle). This analogy provides another interpretation of the term $-\tfrac{1}{2}f^2/\beta$ in (18*c*), because curl $(-\tfrac{1}{2}f^2/\beta, 0, 0) = (0, 0, f)$.)

From (15) we obtain for the perturbation

$$(\partial_t + U\partial_x)q' + v'\partial_y Q = 0, \tag{19a}$$

$$\int dy\,[\partial_t\langle E\rangle + H^2U\langle q'v'\rangle] = 0, \tag{19b}$$

$$\int dy\,[\partial_t\langle M'\rangle + H^2\langle q'v'\rangle] = 0, \tag{19c}$$

where
$$q' = (\partial_x v' - \partial_y u' - Qh')/H, \tag{20a}$$
$$E' = \tfrac{1}{2}[H(u'^2+v'^2)+2h'u'U+g_r h'^2], \tag{20b}$$
$$M' = h'u'. \tag{20c}$$

(Equations (17*b*) and (19*b*) are true even in the presence of arbitrary rigid boundaries with vanishing normal flow. For (17*c*) and (19*c*) to be valid, however, the walls must be along parallels of latitude, i.e. with the same symmetry as the model equations and fundamental flow.)

Note that, unlike in the quasi-geostrophic case, E' is *not* positive-definite.† However, it can be shown that if the flow is subcritical everywhere, i.e. if $U(y)^2 \leqslant g_r H(y)$ is satisfied for all values of y, then $E' \geqslant 0$. Moreover, this can be generalized in the form
$$[U-\alpha]^2 \leqslant g_r H \Rightarrow E' - \alpha M' \geqslant 0, \tag{21}$$
where α is any real constant with dimensions of speed.

Let $\Delta(x,y,t)$ to the $O(\epsilon)$ meridional displacement with respect to the fundamental state, viz
$$v' = (\partial_t + U\partial_x)\Delta. \tag{22}$$

The general solution of (19*a*) has the form
$$q' = F[x-U(y)t, y] - \Delta\,\partial_y Q, \tag{23}$$
where $F(x,y)$ is arbitrary (note that $(\partial_t + U\partial_x)F = 0$). Using (22) we obtain
$$\langle q'v'\rangle = \partial_t\langle F\Delta - \tfrac{1}{2}\Delta^2\partial_y Q\rangle. \tag{24}$$

Since F is bounded in time (e.g. the zonal average of any function of F is time-invariant), the term $\langle F\Delta\rangle$ may be neglected in establishing the stability of the fundamental flow. (For instance, for each eigensolution of the form $(u',v',h') \propto \exp(-i\omega t)$ with $\mathrm{Im}(\omega) \neq 0$, then $\langle q'\rangle = 0$ (for shear instability) requires $F \equiv 0$, whereas for inertial instability F is time-independent.) Combining (19*b*, *c*) and (24), we get
$$\int dy\,\langle (E'-\alpha M') + \tfrac{1}{2}H^2(\alpha - U)\,\partial_y Q\Delta^2\rangle = \text{const.} \tag{25}$$

Therefore, if there exists a value of α such that the left-hand inequality in (21) is satisfied, and $[\alpha - U(y)]\,\partial_y Q(y) \geqslant 0$ for all y, then (25) means that a sum of quadratic and positive-definite functionals of (u',v',h') is time-independent, and consequently the amplitude of the perturbation is bounded. Whence, we can state the following THEOREM:

if there exists any value of α such that
$$[\alpha - U(y)]\,\partial_y Q(y) \geqslant 0 \quad \textit{and} \quad [\alpha - U(y)]^2 \leqslant g_r H(y) \quad \textit{for all } y \tag{26a, b}$$
then the flow is stable to infinitesimal perturbations.

The stability condition found by Miyata (1981) corresponds to that of (26) for the particular case $\alpha = 0$, which in turn reduces to that obtained by Blumen (1970) if $f \equiv 0$. It is shown below and with the example of §3 that the freedom in the choice of the value of α makes (26) a more powerful condition. (Incidentally, the equation

† However, $E'' = \tfrac{1}{2}H(u'^2+v'^2)+\tfrac{1}{2}g_r h'^2$, which is not the $O(\epsilon^2)$ contribution to E, is positive-definite, and $\int dy\,[\partial_t\langle E''\rangle + H\,\partial_y U\langle u'v'\rangle] = 0$ is related to eddy driving through Reynolds stress terms (Griffiths, Killworth & Stern 1982). If $\partial_y Q \neq 0$ then E'' is the lowest-order contribution to $E_0 - P_0 U$ (§4).

(26) for $\alpha \neq 0$ cannot be obtained from the same equation with $\alpha = 0$ via a Galilean transformation, simply because the system (2) is not Galilean invariant.)

Weaker sufficient stability conditions are obtained as corollaries of (26): viz choosing $\alpha = \max(U)$, which results in

$$\partial_y Q \geqslant 0 \quad \text{for all } y \qquad \text{(condition (i))}, \tag{27a}$$

$$\max(U) \leqslant \min[U + (g_r H)^{\frac{1}{2}}] \quad \text{(condition (ii))}; \tag{27b}$$

or making $\alpha = \min(U)$, which yields

$$\partial_y Q \leqslant 0 \quad \text{for all } y \quad \text{(condition (i))}, \tag{28a}$$

$$\min(U) \geqslant \max[U - (g_r H)^{\frac{1}{2}}] \quad \text{(condition (ii))}. \tag{28b}$$

Finally, these two corollaries of (26) may be combined into an even weaker stability condition, using $\max(A+B) \leqslant \max(A) + \max(B)$ and $\min(A+B) \geqslant \min(A) + \min(B)$:

viz $$\partial_y Q \text{ does not change sign} \qquad \text{(condition (i))}, \tag{29a}$$

$$\max(U) - \min(U) \leqslant \min(g_r H)^{\frac{1}{2}} \quad \text{(condition (ii))}; \tag{29b}$$

i.e. if the gradient of potential vorticity does not change sign and the total change of particle velocity is smaller than the minimum phase speed of long gravity waves, then the flow is stable. If the maxima and minima of H and U occur at the same latitudes, as in the example of §3, then (29) is equivalent to (27) or (28) (depending on the sign of $\partial_y Q$).

Conditions (26), (27) and (29) are equivalent for the example of §3. However, in principle, the set (26) is stronger than (27), (28) or (29). For instance, there might be a flow for which, say, $\partial_y Q$ is positive for $y > y^*$ and negative for $y < y^*$. If, in addition, $U(y) \leqslant U(y^*)$ for $y > y^*$ and $U(y) \geqslant U(y^*)$ for $y < y^*$, then (26*a*) is satisfied for $\alpha = U(y^*)$. Consequently, if (26*b*) is also satisfied for the same value of α, then the flow is stable, even though condition (i) is not fulfilled.

The non-divergent case may be formally obtained making $g_r \rightarrow \infty$. Condition (26*b*) is then trivially satisfied and (26*a*) reduces to the theorem of Fjørtoft (1950) (with α equal to the value of U at the inflexion point), or to that of Kuo (1949) and Rayleigh (1880) (with α outside the range of U), since $\partial_y Q \propto \beta - \partial_{yy} U$ in this limit. Notice that for the mid-latitude case, the scaling in (9) implies

$$|U|/c \sim (\beta c/f_0^2)/[1 + (R/L)^2] \leqslant (\beta c/f_0^2),$$

i.e. (29*b*) is satisfied, and thus '$(\partial_y Q)_{\text{qg}}$ does not change sign' is a sufficient condition of stability for westward jets.

Neither (16*a*) nor (16*b*) was needed in order to derive (25), and consequently the set (26), or any of its corollaries, gives sufficient conditions for both shear and inertial stability. For the latter, the more-classical condition (and independent from (26)) is derived as follows: Dropping all terms in which ∂_x appears, the system (15) is easily reduced to

$$\partial_{tt} v' + fQHv' - \partial_{yy}(g_r H v') = 0. \tag{30}$$

Thus for an eigensolution such that $v' \propto \exp(-i\omega t)$ it follows that

$$\omega^2 = \int dy\,[fQH^2|v'|^2 + g_r|\partial_y(Hv')|^2 \int dy\, H|v'|^2, \tag{31}$$

from which we obtain $\omega^2 \geqslant \min[f(f - \partial_y U)]$. Therefore, *if*

$$f(f - \partial_y U) \geqslant 0 \quad \text{for all } y \quad \text{(condition (iii))} \tag{32}$$

then the flow is inertially stable. This condition takes the same form in the tropics and at higher latitudes (see e.g. Dunkerton 1981). (Note that ω^2 is real. For $\omega^2 < 0$, the flow is unstable.)

If $U \equiv 0$, and thus $H = \bar{h}$, the solutions of (30) are longitude-independent *inertial* waves (with eigenfrequencies $\omega^2 = (2n+1)\ \beta c$, $n = 0, 1, 2, \ldots$, in the equatorial β-plane), which is consistent with the suggestion by Busse & Chen (1981 *a*, *b*) of using the name 'inertial instability' for the case (16*b*), and not for the case (16*a*).

If in (30) we approximate $H \approx H(0)$ and $\partial_y U \approx \partial_y U(0)$, then the eigensolutions are parabolic-cylinder functions with argument proportional to $y - \partial_y U/2\beta$, and eigenfrequencies

$$\omega^2 \approx (2n+1)\,\beta(g_r H)^{\frac{1}{2}} - (\tfrac{1}{2}\partial_y U)^2 \quad (n = 0, 1, 2, \ldots) \tag{33}$$

(Dunkerton 1981). The approximation is valid as long as $(2n+1)\,(g_r H)^{\frac{1}{2}}/\beta \ll L^2, L'^2$, where L and L' are the lengthscales for the variations of $\partial_y U$ and H respectively. Even though $\partial_y U(0) \neq 0$ implies that condition (iii) is violated sufficiently close to the equator (no matter how small $|\partial_y U|$ may be), (33) shows that, in order for the flow to be inertially unstable, $\omega^2 < 0$, it must be $(\partial_y U)^2 > 4\beta(g_r H)^{\frac{1}{2}}$. However, for such a strong shear it is not possible to neglect the variations of $H(y)$: from the geostrophic balance (5), and assuming $U(0) = 0$, we get $\beta L'^3 \sim g_r H/\partial_y U$, which is incompatible with $(g_r H)^{\frac{1}{2}}/\beta \ll L'^2$; the approximations used to obtain (33) fail and therefore it is not clear whether the flow might be inertially unstable or not.

In other words, (32) is also a sufficient, but certainly not a necessary, stability condition. In §3 we present some cases for which (32) is violated but the flow is proved to be stable using the set (29). The opposite might also be true, viz a shearly unstable but inertially stable flow, for which (32) is satisfied but not (26).

3. Example: the Gaussian jet

As an example, consider the family of Gaussian jets

$$\frac{H}{\bar{h}} = 1 + \lambda \exp\left[-\frac{y^2}{L^2}\right], \quad \frac{U}{c} = \left(\frac{R_0}{L}\right)^2 \lambda \exp\left[-\frac{y^2}{L^2}\right], \tag{34}$$

where $y = 0$ represents the equator. The parameter λ gives the maximum relative depth change, which occurs at the equator, and L is the e-folding half-width of the jet. (If $\lambda < -1$ there are two fronts at $y^2 = L^2 \ln|\lambda|$; equatorward from these latitudes we define $H, U \equiv 0$. This case is better represented by (35) below with $\lambda = -1$.)

An interesting property of this family of flows is that for $L = R_0$ (i.e. the structure of a Kelvin wave with vanishing zonal wavenumber) $Q = f/\bar{h}$ for all y, and thus $\partial_y Q$ nowhere changes sign, no matter what the value of λ may be. In fact, for $L \geqslant R_0$ and $\lambda \geqslant 0$ (wide westerlies) $\partial_y Q > 0$ for all y. Everywhere in this paper we use the words 'wide' and 'narrow' comparing the width of a jet with that of a Kelvin wave at the same reference latitude.

The parameters of stable flows, as guaranteed by (27) and (32), belong to the dark shaded regions in figure 3; the details of the calculation are given in the appendix. The coordinates in parts (*a*) and (*b*) of that figure are chosen so as to emphasize the parameters of wide and narrow jets respectively (i.e. by plotting $H(0)$ *vs.* L^{-1} or $U(0)$ *vs.* L).

Condition (27*b*) is, in this case, equivalent to choosing $\alpha = 0$ for $\lambda < 0$ and $\alpha = c$ for $\lambda > 0$ in (26). Note that if in (26*b*) one only uses $\alpha = 0$, as in Miyata (1981), then the shear stability of eastward jets cannot be assured.

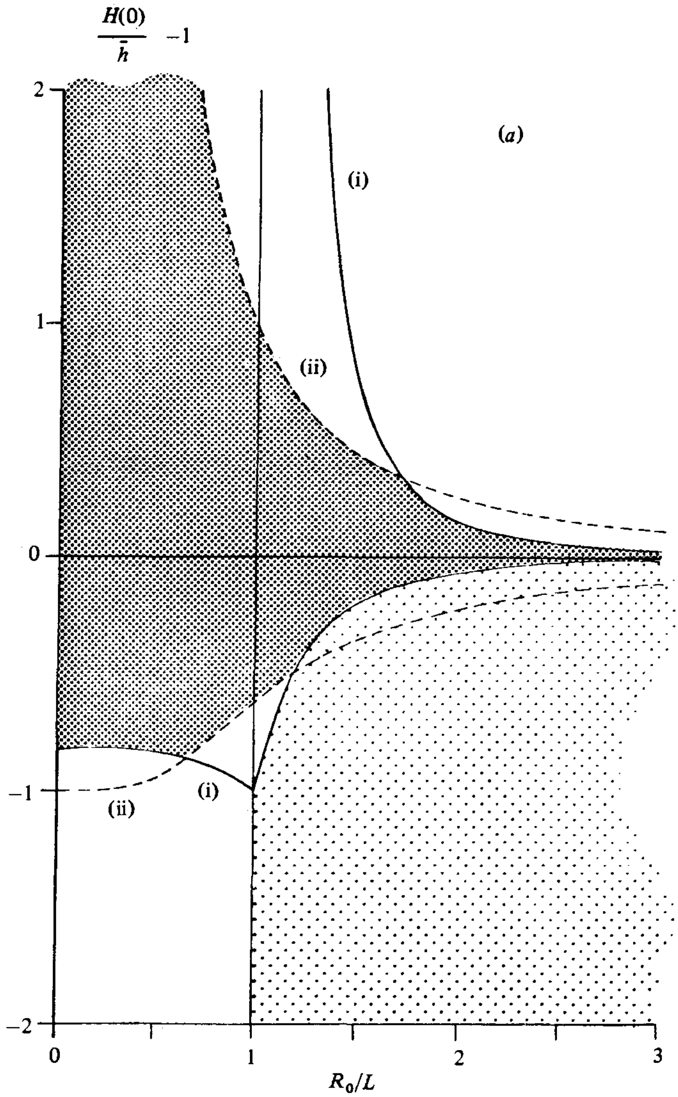

FIGURE 3 (*a*). For caption see facing page.

	$L \ll R_0$	$L = R_0$	$L \gg R_0$
$U(0)$	$\frac{1}{4}e^{\frac{3}{2}}\beta L^2$	c	c
$H(0)/\bar{h}-1$	$\frac{1}{2}e^{\frac{3}{2}}(L/R_0)^4$	1	$(L/R_0)^2$
y^2	$\frac{3}{2}L^2$	—	—
Condition	(i)	(ii)	(ii)
$U(0)$	$-\frac{1}{2}\beta L^2$	$-\frac{1}{2}(\sqrt{5}-1)c$	$-e^{\frac{1}{2}}c^2/\beta L^2$
$H(0)/\bar{h}-1$	$-(L/R_0)^4$	$-\frac{1}{2}(\sqrt{5}-1)$	$\frac{1}{2}e^{\frac{1}{2}}$
y^2	0	—	$\frac{1}{2}L^2$
Condition	(i)–(iii)	(ii)	(i)

TABLE 1. Critical amplitudes beyond which one or more of the stability conditions is violated, and the latitude y where that happens (for conditions (i) or (iii)). Top and bottom panels correspond to westerlies and easterlies respectively. See figure 3.

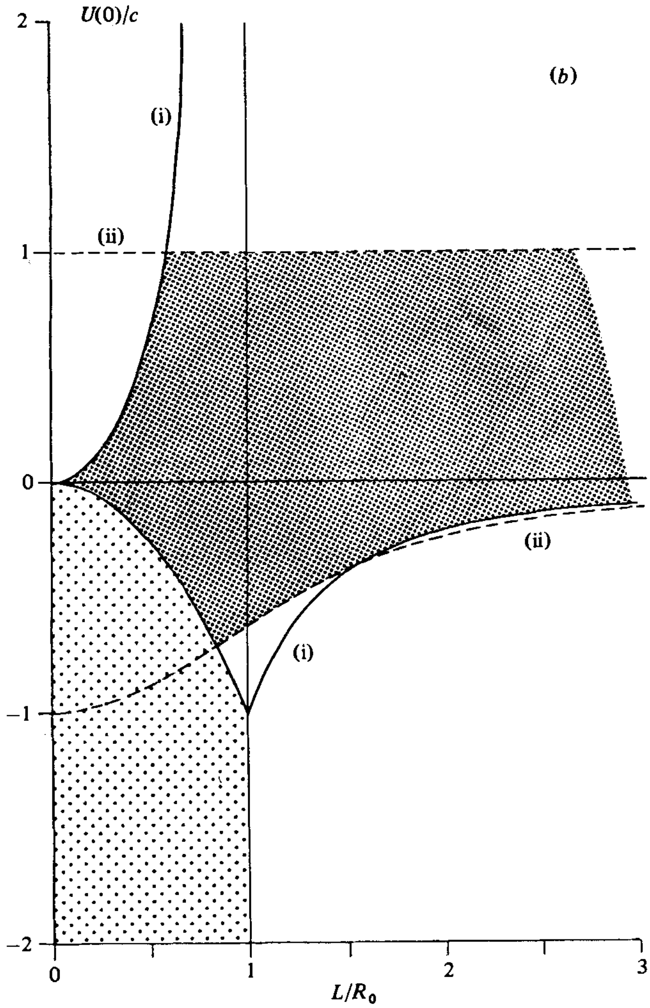

FIGURE 3. Stability of a Gaussian jet centred at the equator. $H(0)$ and $U(0)$ are the equatorial depth and zonal velocity, and L is the e-folding half-width of the jet. For amplitudes in the dark shaded region the flow is stable, as guaranteed by conditions (i) (non-negative gradient of potential vorticity, (27a)) and (ii) (subcriticality, (27b)). Inertial stability is independently guaranteed by condition (iii) (same sign of planetary and potential vorticities, (32)). Conditions (i) and (ii) are violated for amplitudes beyond the solid and dashed lines, respectively. Condition (iii) is violated in the light-shaded region. (The cases with $H(0) < 0$ actually represent a zonal front, with the active layer poleward from it.)

The most restrictive stability conditions are: for narrow westerlies (i); for westerlies wider than $L/R_0 = 0{\cdot}5799\ldots$ (ii); for narrow easterlies (i) and (iii); for easterlies with L/R_0 between $0{\cdot}8409\ldots$ and $1{\cdot}5244\ldots$ (ii); and easterlies wider than $L/R_0 = 1{\cdot}5244\ldots$ (i). The extreme amplitudes for $L \ll R_0$, $L = R_0$, and $L \gg R_0$ are given in table 1. (The prediction of quasi-geostrophic theory, (1) with $f_0 = 0$, coincides with the entry in table 1 corresponding to $L \ll R_0$.) The solid curves in figure 3 show the scaling (12) and (13), as expected, for narrow westerlies and easterlies of any width (except near $L = R_0$).

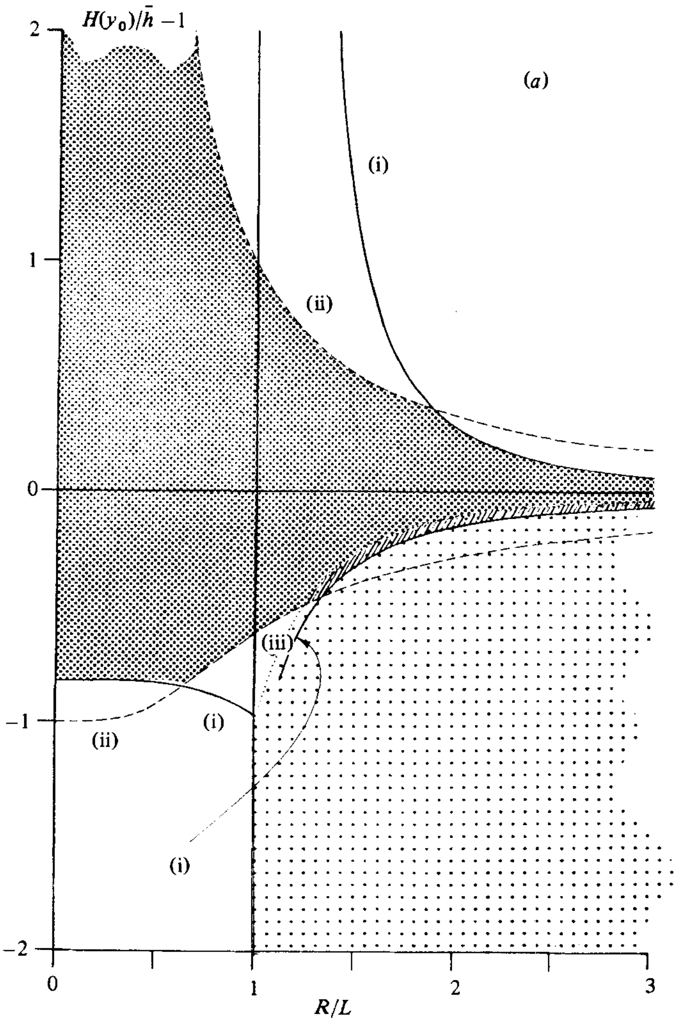

FIGURE 4. (*a*) For caption see facing page.

3.1. *Effect of meridional boundaries: the semi-infinite ocean*

In order to compare the above results with those of extra-equatorial dynamics, we study the stability regions (as defined by conditions (i) and (ii) or (iii) of the jet (34)) but restricting the domain of the model to an infinite semiplane. For a zonal boundary (or a front) at a certain latitude, where the Coriolis parameter and its northward gradient are f_0 and β respectively, we write

$$\frac{H}{\bar{h}} = 1+\phi, \quad \frac{U}{c} = \mu\phi, \quad \phi = \lambda \exp\left[-\frac{\mu(f^2-f_0^2)}{2\beta c}\right] \quad (f \geqslant f_0); \tag{35}$$

the case $f \leqslant -f_0$ is clearly equivalent to this one. (We use $f_0 \geqslant 0$, since for $f_0 < 0$ the stability regions are those of figure 3.) Note that if we redefine y so that $f = \beta y$ and $f_0 = \beta y_0$, then (35) becomes essentially (34). The inflexion point of the flow in (35) is at the latitude where $\mu f^2 = \beta c$, and it therefore belongs to the domain of the model only if $\mu \leqslant \beta c/f_0^2$. For $\mu \ll \beta c/f_0^2$ the jet has the shape of half a Gaussian curve, whereas for $\mu \gg \beta c/f_0^2$ it has essentially an exponential offshore structure.

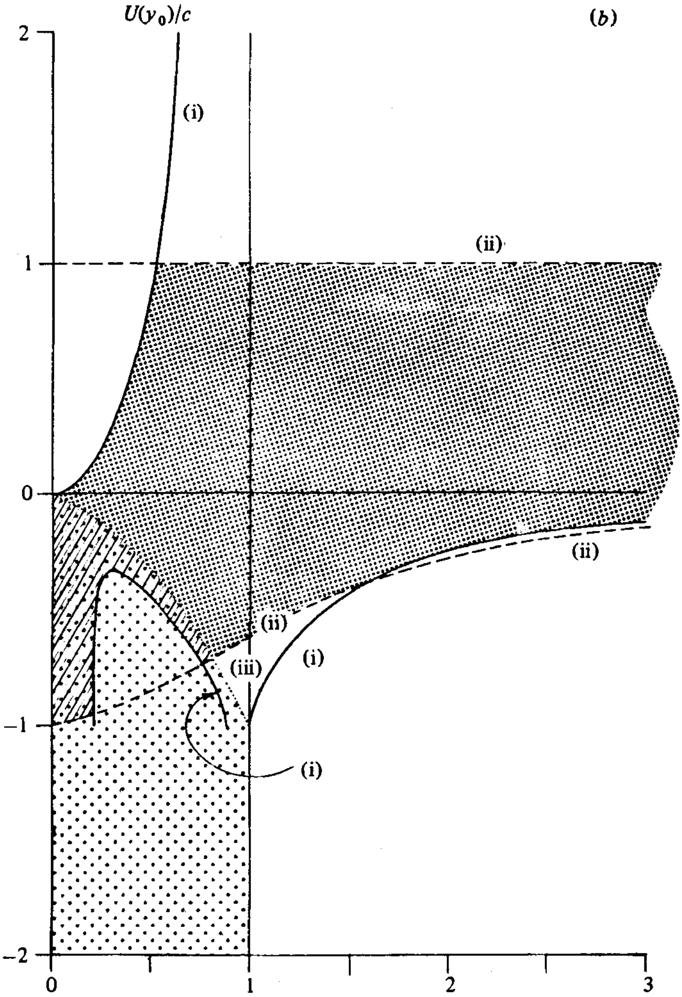

FIGURE 4. As in figure 3, when the model's domain is restricted to the infinite semiplane $y \geqslant y_0$ (or $y \leqslant -y_0$) with $y_0 = 0{\cdot}2R_0$. Flows in the hatched region are stable, even though condition (iii) is violated. The local deformation radius and speed scale are given by, neglecting variations of β with the latitude, $R = 0{\cdot}8198R_0$ and $\frac{1}{2}\beta R^2 = 0{\cdot}6721c$.

The parameter λ represents again the maximum relative depth change, which happens at $f = f_0$, and μ determines the e-folding width of the jet as

$$L = \frac{2c}{f_0\mu + (f_0^2\mu^2 + 2\beta c\mu)^{\frac{1}{2}}}. \tag{36}$$

It follows from this expression that

$$L = R \quad \text{for} \quad \mu = 1 \tag{37a}$$

$$L \sim \frac{c}{f_0\mu} \quad \text{for} \quad \mu \gg \frac{\beta c}{f_0^2} \quad \left(L \ll \frac{f_0}{\beta}\right), \tag{37b}$$

$$L \sim \left(\frac{2c}{\beta\mu}\right)^{\frac{1}{2}} \quad \text{for} \quad \mu \ll \frac{\beta c}{f_0^2} \quad \left(L \gg \frac{f_0}{\beta}\right). \tag{37c}$$

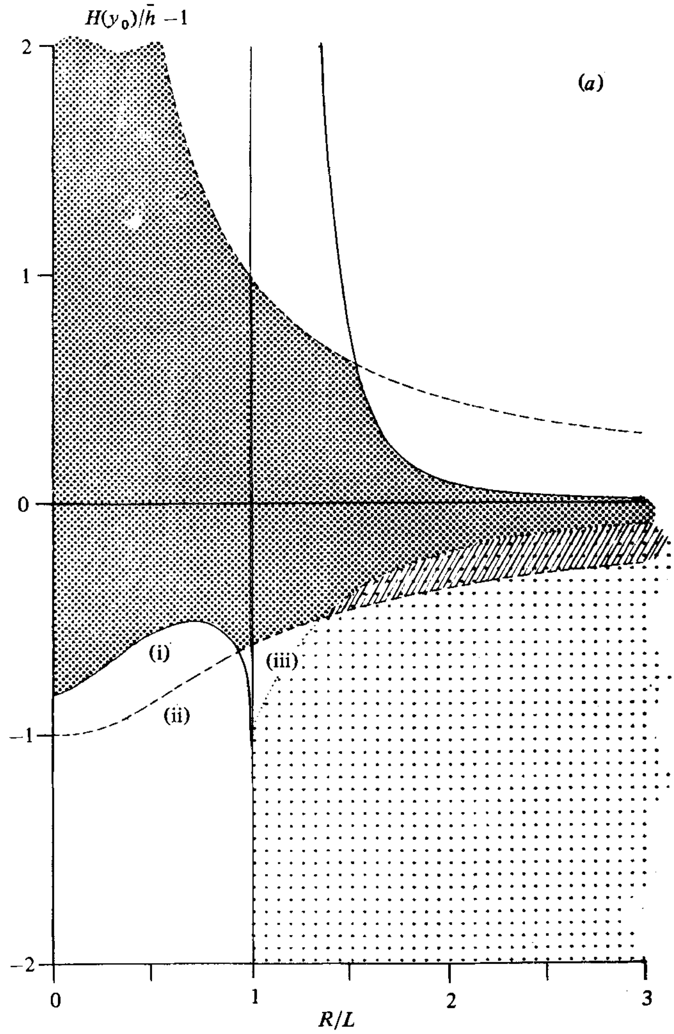

FIGURE 5 (*a*). For caption see facing page.

The stability regions of the flow (35) are presented in figures 4 and 5, for two values of $f_0^2/\beta c$. The details of the calculation are given in the appendix.

The most striking difference with the unbounded case of figure 3 is the appearance (in figure 4) of two gaps, at $L \geqslant 0$ and $L \leqslant R$, where the gradient of potential vorticity is always positive for any negative value of λ (easterlies). If $y_0/R_0 = 1-\sqrt{\frac{1}{2}}$ both gaps merge into one, and thus for

$$y_0/R_0 > 1-\sqrt{\tfrac{1}{2}} \approx 0{\cdot}2929 \tag{38}$$

the gradient of potential vorticity is positive for *any* narrow easterly (as it is for any wide westerly, independently of the value of y_0); see figure 5. The extreme amplitudes for $L \ll R$, $L = R$, and $L \gg R$ are given in table 2. The only difference with the results in table 1 is in the first column: very narrow eastward or westward jets.

A front at the latitude y_0, with the active layer poleward from the front, may be represented by (35) with $\lambda = -1$ ($H(y_0) = 0$). One such front is outside the stability region in figures 4 and 5 (see also table 2).

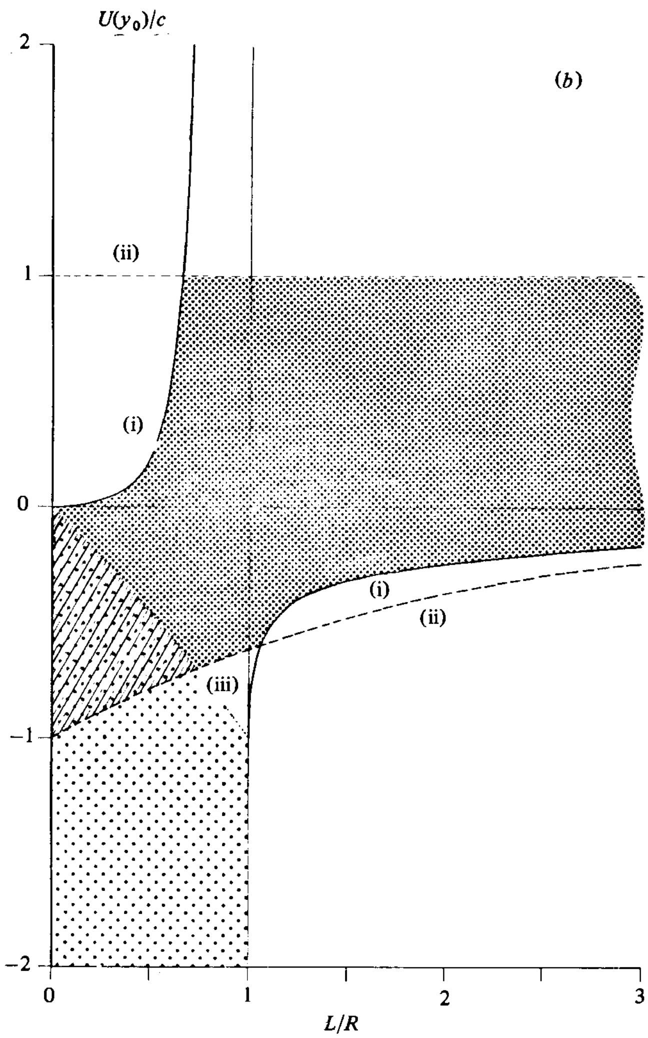

FIGURE 5. As in figure 4, for $y_0 = R_0$. The local deformation radius and speed scale are given by $R = 0{\cdot}4142R_0$ and $\frac{1}{2}\beta R^2 = 0{\cdot}1716c$.

	$L \ll R, f_0/\beta$		$L = R$	$L \gg R, f_0/\beta$
$U(y_0)$	βL^2		c	c
$H(y_0)/\bar{h}-1$	$\beta L^3 f_0/c^2$		1	$(L/R_0)^2$
y^2	y_0^2		—	—
Condition	(i)		(ii)	(ii)
$U(y_0)$	$-Lf_0$	$[-c]$	$-\frac{1}{2}(5^{\frac{1}{2}}-1)\,c$	$-e^{\frac{1}{2}}c^2/\beta L^2$
$H(y_0)/\bar{h}-1$	$-(Lf_0/c)^2$	$[-Lf_0/c]$	$-\frac{1}{2}(5^{\frac{1}{2}}-1)$	$-\frac{1}{2}e^{\frac{1}{2}}$
y^2	y_0^2	[—]	—	$\frac{1}{2}L^2$
Condition	(iii)	[(ii)]	(ii)	(i)

TABLE 2. As in table 1 for the model restricted to $y \geqslant y_0$. See figures 4 and 5. For narrow easterlies, the flow is stable up to the amplitudes in [...], even though condition (iii) is violated at a much smaller amplitude.

Paldor (1982) proved explicitly the stability of a front in the f-plane, $\beta = 0$, for the particular case of constant-potential-vorticity basic flow. This system may be generalized to the β-plane by the case $Q = f/\bar{h}$ (here $\bar{h}$ should be taken as a 'reference' rather than 'mean' depth). The (U, H)-fields are given by (35) with $\lambda = \mu = -1$, and with the active layer equatorward from the fronts, $f^2 \leqslant f_0^2$. (The structure is that of westward-propagating Kelvin mode in the limit of vanishing zonal wavenumber, or 'anti-Kelvin' wave in the curious notation of equatorial dynamicists.) The stability of this flow may easily be proved using (26): since $\partial_y Q = \beta/\bar{h} \geqslant 0$, (26$a$) is satisfied for $\alpha \geqslant \max(U)\,(=c)$, whereas (26$b$) holds for $\alpha = c$, and thus the flow is stable.

4. Stability and energy–pseudomomentum conservation

The system (2) has only two integrals of motion that are, to lowest order, quadratic in $[u, v, h-\bar{h}]$, viz the total energy (17b) and the total zonal pseudomomentum (Ripa 1982)

$$\int dy\,\langle P\rangle = \text{const}, \quad P = u(h-\bar{h}) - \frac{h(q\bar{h}-f)^2}{2\beta}. \tag{39a, b}$$

For the fundamental flow $[u, v, h-\bar{h}] = [U, 0, H-\bar{h}]$ in (34) it can be shown that

$$\int dy\,E = \tfrac{1}{2}Nc\left[1 + (1 + (\tfrac{2}{3})^{\frac{1}{2}}\lambda)\left(\frac{R_0}{L}\right)^4\right], \tag{40a}$$

$$\int dy\,P = N\left(\frac{R_0}{L}\right)^2\left[1 - \frac{I(\lambda)\,(R_0^4 - L^4)^2}{2R_0^4 L^4}\right], \tag{40b}$$

where $N = \lambda^2 cL\bar{h}(\tfrac{1}{2}\pi)^{\frac{1}{2}}$, and

$$I(\lambda) = \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} dx\,\frac{x^2\exp(-x^2)}{1+\lambda\exp(-\tfrac{1}{2}x^2)}. \tag{41}$$

This integral is finite for any $\lambda \geqslant -1$, and it may be calculated, for $|\lambda| < 1$, expanding the right-hand side of (41) in powers of λ (see figure 6).

The eigenvectors $[u_a, v_a, h_a]$ that result from linearizing (2) and substituting ∂_t by $-i\omega$ (viz (15) for $U = 0$, with H and QH replaced by $\bar{h}$ and f) span an orthogonal and complete basis, which may be used to make the expansion

$$[u, v, h-\bar{h}] = \Sigma\, Z_a(t)\,[u_a(\mathbf{x}), v_a(\mathbf{x}), h_a(\mathbf{x})], \tag{42}$$

where the label a stands for the zonal wavenumber k, the meridional quantum number n, and a discrete index with (at most) three values (Ripa 1982).† An interesting property of this expansion is that the quadratic part of total E and P have a diagonal representation, viz

$$\int dy\,\langle E\rangle = \Sigma\,|Z_a|^2 + O(Z^3), \quad \int dy\,\langle P\rangle = \Sigma\, s_a|Z_a|^2 + O(Z^3). \tag{43a, b}$$

Here s_a is the slowness of the ath expansion mode, and it satisfies $s_a c = 1$ if a is a Kelvin mode and $s_a c < 1$ if it is not.

For quasi-geostrophic flow at mid-latitudes there are no $O(Z^3)$ terms in (43), and

† The symbol Σ includes an integration in zonal wavenumber. The 'summations' in (43) and (67) run over physically different states, i.e. only one mode for each conjugate pair, with parameters (ω, k) and $(-\omega, -k)$. Depending on the problem, $\int dx$ should be used instead of $\langle\ \rangle$. For the details, see Ripa (1981a, 1982).

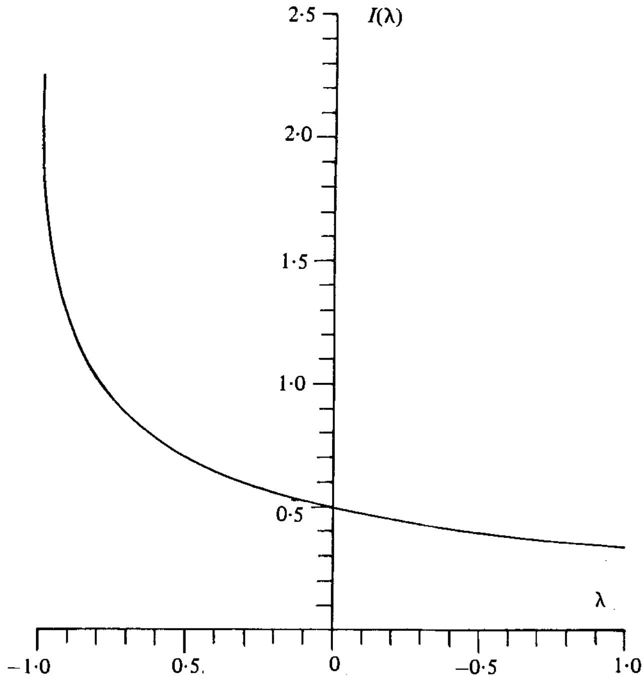

FIGURE 6. Integral (41) used for the evaluation of the total pseudomomentum of the fundamental flow.

thus $\int dy\,\langle P\rangle/\int dy\,\langle E\rangle$ represents an energy-weighted average slowness, which is conserved in virtue of (17*b*) and (39*a*) (Ripa 1981*a*). As a consequence, if the fundamental flow is unstable, the modes in the expansion (42) of any growing perturbation must have values of s_a both larger and smaller than $\int dy\,\langle P\rangle/\int dy\,\langle E\rangle$. (For the case of non-divergent flow this is no more than a re-phrasing of Fjørtoft's (1950) theorem.) This result has been used for the problem of the stability of a finite-amplitude Rossby wave (Gill 1974; Ripa 1981*a*), including divergence effects.

The normal modes $[u_a(\mathbf{x}), v_a(\mathbf{x}), h_a(\mathbf{x})]$ for the case of quasi-geostrophic flow at mid-latitudes are the position-dependent parts of Rossby waves, which have $sc < -f_0^2/\beta c\,(\ll -1)$. Therefore, a fundamental flow for which

$$c\int dy\,\langle P\rangle/\int dy\,\langle E\rangle = O(1)$$

is stable to quasi-geostrophic perturbations. In other words, in order for that flow to be unstable it must have such a large amplitude that the quasi-geostrophic approximation is not valid. This is, for instance, the case of the flow (35) with $\mu = 1$, for which $Pc \equiv E + O(\lambda)$.

For the fundamental flow (34), the expansion (42) only involves the longitude-independent geostrophic modes, with meridional quantum numbers $n = -1, 1, 3, 5, \ldots$ (Kelvin and ultralong Rossby 'waves'). (The modes with $n = 2, 4, 6, \ldots$ are excluded owing to the symmetry of the flow, $U(y) = U(-y)$.) Since for these modes $s_a c = -2n-1$, an idea of the energy spectrum of (34) is given by an 'average' value of n, defined by

$$\frac{\int dy\,\langle P\rangle\,c}{\int dy\,\langle E\rangle} = -2\bar{n}-1+O(\lambda). \tag{44}$$

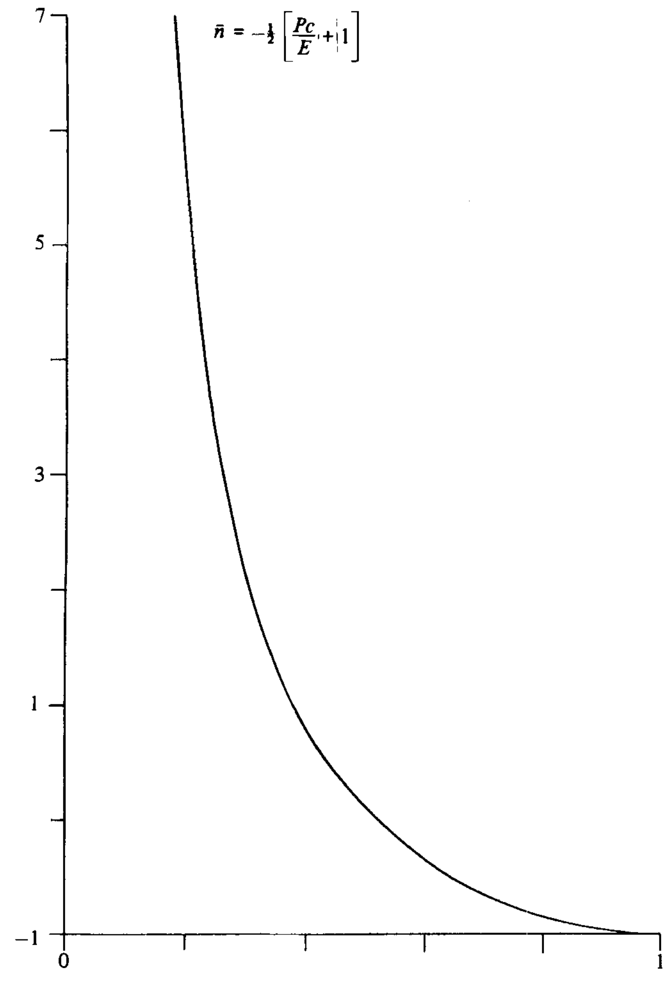

FIGURE 7. Mean meridional quantum number for the modal decomposition of the fundamental flow, as defined in (44).

where both integrals are taken from (40). Thus, using $\lambda = 0$ in both expressions between square brackets ($I(0) = \frac{1}{2}$), it follows that

$$\bar{n} = [L^4 + R_0^4]^{-1}\left[\frac{(L^4 - R_0^4)^2}{4R_0^2 L^2} - R_0^2 L^2\right] - \tfrac{1}{2}, \tag{45}$$

which satisfies $\bar{n} \geqslant -1$, as it should. See figure 7.

Now, if the $O(Z^3)$ terms in (43) are negligible and the fundamental flow is unstable, then in the expansion (42) of any growing perturbation there must be components with $s_a \geqslant -(2\bar{n}+1)/c$ and components with $s_a \leqslant -(2\bar{n}+1)/c$ (or else $s_a c = -(2\bar{n}+1)$ for all modes). Then for $L = R_0$, $\bar{n} = -1$, the jet would be stable for any $\lambda \geqslant -1$, because there are no components with $s_a c > 1$. (The possibility that all components have $s_a c = 1$, Kelvin modes, is discussed in Ripa (1982).) This is the case in which the fundamental flow is a longitude-independent Kelvin mode, which is stable *at least* for values of λ ($\equiv U(0)/c$) between $-\frac{1}{2}(\sqrt{5}-1)$ ($\approx -0{\cdot}6180$) and 1 (see table 1). Even though the flow may not be strictly stable (for any λ) when $L = R_0$, the minimum amplitude required for instability is quite large (100 % or 62 % depth change, see figure 3); in fact, large enough to make the $O(Z^3)$ terms in (43) important. Another 'prediction' of energy–pseudomomentum conservation is that, for an unstable flow with $L \sim R_0$, the Kelvin mode should be an energetic component of any growing perturbation.

For $L \ll R_0$ or $L \gg R_0$, on the other hand, $\bar{n} \gg 1$, and thus components with $s_a c = O(1)$ (inertia–gravity and Kelvin modes) should not gain much energy from the decay of an unstable jet.

4.1. *Pseudo-energy and momentum relative to the fundamental flow*

The zonal pseudomomentum density (39*b*) has the form

$$P = M - \frac{hq^2\bar{h}^2}{2\beta} + \text{time-independent and divergent terms} \tag{46}$$

(the latter are given by $\bar{h}[f^2 + \partial_x(fv) - \partial_y(fu)]$. Total P is conserved because the integral of M is time-independent, (17*c*), and so is the integral of $hF(q)$, with F an arbitrary function, by virtue of (2*c*) and (17*a*). (These conservation laws are related to the invariance of the system to zonal translations (Ripa 1981*b*).) The particular form of $F(q)$ in (47) was chosen so as to eliminate the linear term of M, $u\bar{h} - (h-\bar{h})f^2/2\beta$, making total P a quadratic, to lowest order, functional of $[u, v, h-\bar{h}]$ (i.e. quadratic in the departure from the reference state $[u = 0,\ v = 0,\ h = \bar{h}]$). This method may be used to construct pseudo-energy and momentum densities E_0 and P_0, which are quadratic in the departure from the new reference state $[y = U(y),\ v = 0,\ h = H(y)]$. Defining

$$u = U + u', \quad v = v', \quad h = H + h', \tag{47}$$

where the fields (u', v', h') are of *any* amplitude [i.e. there are no $O(\epsilon^2)$ terms as in (14)), and defining q' by (20*a*), as before, it can be shown that

$$q = Q + q'H/h \tag{48}$$

exactly. Therefore, for any function $F(q)$,

$$\int dy\, \langle hF(q) - HF(Q)\rangle = \int dy \left\langle h'[F(Q) - QF'(Q)] + u'[\partial_y QF''(Q)] + \frac{q'^2}{2h}F''(Q)\,H^2 + O\left(\frac{q'^3}{h^2}\right)\right\rangle = \text{const} \tag{49}$$

(the zeroth-order term $HF(Q)$ is clearly time-independent, and is subtracted for convenience). Let us seek a function $F(q)$ such that

$$P_0 \equiv M - hF(q) \tag{50}$$

is, to lowest order, quadratic in (u', v', h'). Using

$$M = (U - f^2/2\beta)\,H + (U - f^2/2\beta)\,h' + u'H + u'h'$$

and (49), the linear terms in $\int dy\,\langle P_0\rangle$ are cancelled if and only if

$$QF'(Q) - F(Q) = \frac{f^2}{2\beta} - U, \quad (\partial_y Q)F''(Q) = H. \tag{51a, b}$$

Taking the y-derivative of (51*a*), we obtain (51*b*), and thus the only condition for the existence of F is that the right-hand side of (51*a*) be a function of Q (i.e. that there exists the function $y = y(Q)$). Since $d(f^2/2\beta - U)/dQ = QH/\partial_y Q$,

$$\partial_y Q \neq 0 \tag{52}$$

must hold in order for $F(Q)$ to exist. In the rest of this section we assume that this equation does hold.

Similarly, defining
$$E_0 = E + hG(q), \tag{53}$$
and choosing G so that the linear terms, in (u', v', h'), of $\int dy\,\langle E_0\rangle$ vanish, it follows that G must satisfy
$$QG'(Q) - G(Q) = g_r H + \tfrac{1}{2}U^2, \quad (\partial_y Q)G''(Q) = -UH. \tag{54a, b}$$
Once again, (54b) is a consequence of (54a), and (52) must be satisfied for G to exist, since $d(g_r H + \frac{1}{2}U^2)/dQ = -QUH/\partial_y Q$. (Incidentally, the relationship giving $g_r H + \frac{1}{2}U^2$ as a function of Q is that used by Hughes (1979, 1981) in the study of the stability of the equatorial undercurrent.)

In summary, and using (49), if (52) is satisfied, then F and G exist such that total P_0 and E_0 are conserved, with
$$P_0 = u'h' - \tfrac{1}{2}q'^2 H^2/\partial_y Q + O(u', v', h')^3, \tag{55a}$$
$$E_0 = P_0 U + \tfrac{1}{2}(Hu'^2 + Hv'^2 + g_r H'^2) + O(u', v', h')^3 \tag{55b}$$
(plus exact derivatives and time-independent terms). Note that the lowest-order contribution, in (u', v', h'), to E and M is not given by E' and M', but, rather, by $O(\epsilon)$ terms (see (18) and (20)), which do not average out for longitude-independent perturbations. However, (25), used to derive the stability conditions, is no more than the lowest-order contribution to
$$\int dy\,\langle E_0 - \alpha P_0\rangle = \text{const!} \tag{56}$$
This result is important because it shows that (25) is not an artifact of the linearization in (14), but, rather, the lowest-order contribution of a true, nonlinear, conservation law. (There are examples in the literature of integrals of motion deduced for a linearized system which are meaningless for the fully nonlinear problem.) The derivation of stability conditions from integral conservation laws is a powerful method; e.g. because the results are not restricted to normal-mode infinitesimal disturbances but, rather, they are valid for any small perturbation. Similar techniques have been used by other authors in the study of problems more general than the one considered here (i.e., three-dimensional and/or non-parallel), but for systems somewhat simpler than the one in this paper (viz only the stream function is needed because the flow is plane).

Drazin & Howard (1966) used the equivalent of $\int E_0\,dx\,dy = \text{const}$ to derive the stability condition for non-parallel, plane and non-rotating flow; see their equation (2.44). Blumen (1968) generalized this result, including rotation within the framework of the quasi-geostrophic approximation, using Arnold's (1965) variational principle. (This principle may be cast, for the model of this paper, in the following way. If a function $G(q)$ may be found such that $I\{u,v,h\} \equiv \int\{E + hG(q)\}\,dx\,dy$ is a local minimum at $\{u,v,h\}$ equal to some stationary flow, then that flow is stable because I is an integral of motion. For the particular case of zonal flow, the conditions for I to be a minimum are (26a, b) with $\alpha = 0$, and $E + hG(q) \equiv E_0$.)

The stability of three-dimensional parallel flow has been studied by Bretherton (1966) and Blumen (1978), in the context of the quasi-geostrophic approximation. Bretherton's equation (13) is the equivalent of the α-term (momentum) in (25) here; in his case the integral of $h'u'$ is neglected. Blumen's equations (11) and (14) correspond to $\int E_0\,dx\,dy = \text{const}$ and $\int P_0\,dx\,dy = \text{const}$.

We conclude this section by presenting a self-adjoint problem, whose eigensolutions may be used to expand the fields (u', v', h'), and such that the integrals of $E_0 - P_0 U$

and P_0 have a diagonal representation, as E and P have in (43). The eigenvalue problem is

$$\omega_a u_a - iQHv_a + g_r i\,\partial_x h_a = 0, \tag{57a}$$

$$\omega_a v_a + iQHu_a + g_r i\,\partial_y h_a = 0, \tag{57b}$$

$$\omega_a h_a + i\,\partial_x(u_a H) + i\,\partial_y(v_a H) = 0, \tag{57c}$$

with the 'boundary' condition that the fields be bounded at infinity. The set (57) does *not* constitute the modal problem of (15) for the flow $U(y)$; e.g. the latter is not self-adjoint (otherwise all flows would be stable to normal-mode perturbations). Taking the curl of (57*a*, *b*), we obtain

$$\omega_a q_a + v_a i\,\partial_y Q = 0, \quad q_a = \frac{\partial_x v_a - \partial_y u_a - Qh_a}{H}. \tag{58a, b}$$

Multiplying (57*a*, *b* and *c*) by $Hu_b{}^*$, $Hv_b{}^*$ and $g_r h_b{}^*$, subtracting the complex conjugate of the result after exchanging a and b, and integrating, we obtain

$$(\omega_a - \omega_b{}^*) \int dx\,dy\,[Hu_b{}^* u_a + Hv_b{}^* v_a + g_r h_b{}^* h_a] = 0. \tag{59}$$

For $a = b$, the integral is positive, and thus any eigenvalue ω_a is real; it then follows that the integral vanishes if $\omega_a \neq \omega_b$. (These are, of course, general consequences of the operator in (57) being Hermitian, for a scalar product defined by the integral in (59).) Since different eigenstates corresponding to the same eigenvalue may be made orthogonal, we have

$$\int dx\,dy\,[Hu_b{}^* u_a + Hv_b{}^* v_a + g_r h_b{}^* h_a] = \delta(a,b), \tag{60}$$

where $\delta(a,b) = 0$ if $a \neq b$. In a similar way, from (57*a*, *c*) and (58*a*) it follows that

$$\omega_a \int dx\,dy \left[u_b{}^* h_a + h_b{}^* u_a - \frac{H^2}{\partial_y Q} q_b{}^* q_a\right] = -i\int dx\,dy\,[Hu_b{}^*\partial_x u_a + Hv_b{}^*\partial_x v_a + g_r h_b{}^*\partial_x h_a]. \tag{61}$$

Therefore, making

$$(u_a, v_a, h_a) \propto \exp{(is_a \omega_a x)}, \tag{62}$$

and using (60), (61) reduces to

$$\int dx\,dy \left[u_b{}^* h_a + h_b{}^* u_a - \frac{H^2}{\partial_y Q} q_b{}^* q_a\right] = s_a \delta(a,b). \tag{63}$$

The basis (u_a, v_a, h_a) is presumably complete, because the problem (57) is self-adjoint, and thus it can be used to make the expansion

$$[u', v', h'] = \Sigma\, Z_a(t)\,[u_a(\mathbf{x}), v_a(\mathbf{x}), h_a(\mathbf{x})], \tag{64}$$

for any field (u', v', h'). Normalizing Σ so that $\Sigma\,\delta(a,b)\,T(b) = T(a)$ for any T, (60) may be used to calculate the expansion amplitudes as

$$Z_a(t) = \int dx\,dy\,[Hu_a{}^* u' + Hv_a{}^* v' + g_r h_a{}^* h']. \tag{65}$$

Replacing this expansion in the integrals of (55), and using (60) and (63), we obtain

$$\int P_0\,dx\,dy = \int dx\,dy \left[u'h' - \frac{q'^2 H^2}{2\partial_y Q}\right] + O(u', v', h')^3 = \Sigma\, s_a |Z_a|^2 + O(Z^3) = \text{const}, \tag{66a}$$

$$\int (E_0 - P_0 U)\,dx\,dy = \int dx\,dy\,\tfrac{1}{2}[Hu'^2 + g_r h'^2] + O(u', v', h')^3 = \Sigma\,|Z_a|^2 + O(Z^3) = \text{const}. \tag{66b}$$

In summary, the integrals of P_0 and E_0 are constants of motion but are not positive-definite; the integral of $E_0 - P_0 U$ is positive-definite but not constant (see the footnote on p. 469); and if there exists a value of α such that the integral of $E_0 - \alpha P_0$ is positive-definite, then the flow is stable.

5. Stability conditions on the sphere

As mentioned in §1, the main weakness of the model in this paper is its vertical structure. The β-plane was used just for simplicity, and in this section we show how all the above results are easily generalized to a truly spherical geometry.

Let (x, y) be, for the moment, any pair of orthogonal curvilinear coordinates in a sphere with radius equal to a. It is convenient to use locally isotropic coordinates, i.e. such that the differential displacement and horizontal gradient operator have the form

$$d\mathbf{r} = \gamma(dx, dy), \quad \nabla = \gamma^{-1}(\partial_x, \partial_y); \tag{67 a, b}$$

here $\gamma(x, y)$ is a geometric coefficient that must satisfy

$$(\partial_{xx} + \partial_{yy}) \ln \gamma + \frac{\gamma^2}{a^2} = 0 \tag{68}$$

in order for the cross-derivatives of the unit vectors to be equal (see for instance appendix 2 of Batchelor 1967).

The equations of motion take the form

$$D_t u - [f + \gamma^{-2}(v\,\partial_x \gamma - u\,\partial_y \gamma)]\,v + g_r \gamma^{-1} \partial_x h = 0, \tag{69a}$$

$$D_t v + [f + \gamma^{-2}(v\,\partial_x \gamma - u\,\partial_y \gamma)]\,u + g_r \gamma^{-1} \partial_y h = 0, \tag{69b}$$

$$D_t h + h\gamma^{-2}[\partial_x(\gamma u) + \partial_y(\gamma u)] = 0, \tag{69c}$$

where

$$D_t = \partial_t + u\gamma^{-1}\,\partial_x + v\gamma^{-1}\,\partial_y. \tag{70}$$

Alternatively, the set (69) may be written as

$$\partial_t u - qhv + \gamma^{-1}\,\partial_x(g_r h + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2) = 0, \tag{69a$'$}$$

$$\partial_t v + qhu + \gamma^{-1}\,\partial_y(g_r h + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2) = 0, \tag{69b$'$}$$

$$\partial_t h + \gamma^{-2}[\partial_x(\gamma u h) + \partial_y(\gamma v h)] = 0, \tag{69c$'$}$$

where

$$q = [f + \gamma^{-2}\,\partial_x(\gamma v) - \gamma^{-2}\,\partial_y(\gamma u)]/h. \tag{71}$$

For an extrapolar zonal current, it is convenient to use a Mercator projection, for which longitude and latitude are given by

$$\phi = \frac{x}{a}, \quad \theta = \arctan \sinh \frac{y}{a}, \tag{72}$$

and the geometric coefficient is

$$\gamma = \operatorname{sech} \frac{y}{a} \equiv \cos \theta \tag{73}$$

(and thus $\gamma^{-1}\,\partial_y = a^{-1}\,\partial/\partial\theta$).† The Coriolis parameter is given by

$$f = 2\Omega \tanh \frac{y}{a}, \tag{74}$$

where $2\pi/\Omega$ is the rotation period (86164 s for the Earth).

† For a circumpolar current, it is better to use the (say, South) polar projection, for which $[x, y] = 2a \tan(\frac{1}{4}\pi + \frac{1}{2}\theta)[\cos\phi, \sin\phi]$, and $\gamma = 4a^2/(4a^2 + x^2 + y^2) \equiv \frac{1}{2}(1 - \sin\theta)$. The results of this section can obviously also be derived using ordinary spherical coordinates, which, however, are not isotropic.

Taking the curl of ($69a$, b)′ it follows that ($17a$) still holds, with the material derivative given by (70). Equations ($17b$, c) are replaced by

$$\int dy\,\gamma^2\langle E\rangle = \text{const}, \quad \int dy\,\gamma^2\langle M\cos\theta\rangle = \text{const}. \qquad (75a, b)$$

The factor γ^2 is here just to make $dx\,dy\,\gamma^2$ the differential of area on the sphere (see ($67a$)), but the factor $\cos\theta$ (which happens to be equal to γ) in ($75b$) is quite a different matter. It makes

$$h\cos\theta\left[u-\frac{f^2}{2\beta}\right] = h\cos\theta\,[u+\Omega a\cos\theta]-\Omega ah. \qquad (76)$$

The integral of Ωah, which is proportional to the total volume, is independently conserved, in virtue of ($69c$); whereas the first term on the right-hand side is the *angular momentum*, because $u+\Omega a\cos\theta$ is the total zonal velocity and $a\cos\theta$ is the distance to the rotation axis. Thus zonal-momentum conservation on the β-plane is an approximation to angular-momentum conservation on the sphere; a similar thing applies to the pseudomomenta. Zonal-momentum M and pseudomomentum P conservation on the β-plane are related to invariance of the system under uniform infinitesimal translations in x (Ripa 1981b); similarly, angular-momentum ($M\cos\theta$) and pseudomomentum ($P\cos\theta$) conservation on the sphere are related to invariance of the system under infinitesimal rotations around the axis.

The geostrophic balance and potential vorticity of the fundamental flow are

$$(f+a^{-1}U\tan\theta)\,U+g_{\mathrm{r}}\frac{\partial H}{\partial\theta} = 0, \qquad (77a)$$

$$Q = \left[f-a^{-1}\sec\theta\frac{\partial(U\cos\theta)}{\partial\theta}\right]\Big/H. \qquad (77b)$$

The equations of motion for the perturbation take the form

$$(\partial_t+U\gamma^{-1}\partial_x)\,u'-QHv'+g_{\mathrm{r}}\gamma^{-1}\partial_x h' = 0, \qquad (78a)$$

$$(\partial_t+U\gamma^{-1}\partial_x)\,v'+[f+a^{-1}2U\tan\theta]\,u'+g_{\mathrm{r}}\gamma^{-1}\partial_y h' = 0, \qquad (78b)$$

$$(\partial_t+U\gamma^{-1}\partial_x)\,h'+\gamma^{-2}[\partial_x(u'H\gamma)+\partial_y(v'H\gamma)] = 0. \qquad (78c)$$

From these equations one deduces

$$(\partial_t+U\gamma^{-1}\partial_x)\,q'+v'\gamma^{-1}\partial_y Q = 0, \qquad (79a)$$

$$\int dy\,\gamma^2\,[\partial_t\langle E'\rangle+H^2U\langle q'v'\rangle] = 0, \qquad (79b)$$

$$\int dy\,\gamma^2\,[\partial_t\langle M'\rangle+H^2\langle q'v'\rangle]\cos\theta = 0, \qquad (79c)$$

where

$$q' = [\gamma^{-2}\partial_x(\gamma v')-\gamma^{-2}\partial_y(\gamma u')-Qh']/H, \qquad (80)$$

but E' and M' are the same as before, ($20b$) and ($20c$). E' is not positive-definite, but it may be shown that if there exists a real value of α such that

$$[U-\alpha\cos\theta]^2 \leqslant g_{\mathrm{r}}H \Rightarrow E'-\alpha M'\cos\theta \geqslant 0. \qquad (81)$$

Finally, defining, as before, Δ to be the $O(\epsilon)$ meridional displacement from the fundamental flow, and using (79), we obtain

$$\int\langle E'-\alpha M'\cos\theta+\tfrac{1}{2}H^2(\alpha\cos\theta-U)\,\gamma^{-1}\partial_y Q\Delta^2\rangle = \text{const}. \qquad (82)$$

Whence, *if there exists any value of* α *such that*

$$[\alpha\cos\theta-U(y)]\,\partial_y Q(y) \geqslant 0 \quad \textit{and} \quad [\alpha\cos\theta-U(y)]^2 \leqslant g_{\mathrm{r}}H(y), \qquad (83a, b)$$

then the flow is stable.

Weaker sufficient stability conditions are obtained as corollaries of this one: viz choosing $\alpha = \max\,(U \sec\theta)$, which results in

$$\gamma^{-1}\,\partial_y Q \geqslant 0 \quad \text{for all } y \quad \text{(condition (i))}, \tag{84a}$$

$$\max\,(U\sec\theta) \leqslant \min\,\{[U+(g_r H)^{\frac{1}{2}}]\sec\theta\} \quad \text{(condition (ii))}; \tag{84b}$$

or making $\alpha = \min\,(U\sec\theta)$, which yields

$$\gamma^{-1}\,\partial_y Q \leqslant 0 \quad \text{for all } y \quad \text{(condition (i))}, \tag{85a}$$

$$\min\,(U\sec\theta) \geqslant \max\,\{[U-(g_r H)^{\frac{1}{2}}]\sec\theta\} \quad \text{(condition (ii))}. \tag{85b}$$

Finally, these two corollaries of (83) may be combined into an even weaker stability condition (using $\max(A+B) \leqslant \max(A)+\max(B)$ and $\min(A+B) \geqslant \min(A)+\min(B)$):

viz
$$\gamma^{-1}\,\partial_y Q \text{ does not change sign} \quad \text{(condition (i))}, \tag{86a}$$

$$\max\,(U\sec\theta)-\min\,(U\sec\theta) \leqslant \min\,[(g_r H)^{\frac{1}{2}}\sec\theta] \quad \text{(condition (ii))}. \tag{86b}$$

As shown in §4, for the case of the β-plane, if the gradient of potential vorticity does not change sign, then a pseudo-energy E_0 and an angular pseudomomentum $P_0\cos\theta$ may be defined which are, to lowest order, quadratic in the deviation from the fundamental state. It then follows that (82) is the lowest-order contribution to the exact law $\int dy\,\gamma^2\langle E_0-\alpha P_0\cos\theta\rangle = \text{const.}$

Finally, for longitude-independent perturbations it follows that

$$\omega^2 \geqslant \min\left[(f+2a^{-1}U\tan\theta)\left(f-a^{-1}\frac{\partial(U\cos\theta)}{\partial\theta}\sec\theta\right)\right], \tag{87}$$

and thus another sufficient condition for inertial stability is that the expression between between square brackets be non-negative everywhere.

6. Discussion

The two conditions (i) and (ii) described in the abstract of this paper guarantee the stability of a zonal one-layer flow on the β-plane: on the sphere, 'velocity', 'momentum' and 'pseudomomentum', should be changed to 'angular velocity', 'angular momentum' and 'angular pseudomomentum'. For the example of the Gaussian equatorial jets, the more restrictive condition may be either (i) or (ii), depending on the width and direction of the flow. Thus for very narrow jets flowing in either direction and very wide westward jets, condition (i) is violated at smaller amplitudes than (ii) is.

On the other hand, consider the case of a westward jet with the shape of a longitude-independent Kelvin wave. The potential vorticity is equal to the planetary one divided by the mean depth, and thus condition (26a) with $\alpha = 0$, $-U\,\partial_y Q \geqslant 0$ is satisfied for any strength of the jet. For the flow to be unstable, then, condition (26b) must be violated; namely, it must be supercritical in some region (which turns out to be a neighbourhood of the equator), so that the perturbation energy E' is not positive-definite. Moreover, the conserved pseudo-energy $E_0\,(\sim E'-(UH^2/\partial_y Q)\,q'^2 > E')$ must not be positive-definite. (The required amplitude is

$$U(0)/c < -\tfrac{1}{2}(\surd 5-1) \approx -0{\cdot}6180.)$$

Cairns (1979) has shown that some instabilities of parallel flows are related to the existence of perturbation waves with negative energy, i.e. that lower the total energy of the system (a well-known phenomenon in plasma physics). (More specifically, the onset of the instability is at the point where two waves with energies of opposite signs have equal wavenumbers and frequencies.) Ripa & Marinone (1983) show that the equatorial jet discussed in this paragraph is indeed unstable (by actually solving the eigenvalue problem); the energetics of the problem and its relationship with that of Cairns (1979) are discussed by Marinone & Ripa (1983). (In both cases, the term responsible for a negative energy is $\langle Uu'h' \rangle$; for both the kinetic and potential energies in Cairns' case, but for only the kinetic energy in the problem of this paper, because in a system that is Lagrangian in the vertical (isopycnal vertical coordinate), potential energy is *exactly* quadratic (Ripa 1981*b*).)

Similarly, for an eastward Gaussian jet with the width of a Kelvin wave or wider, the gradient of potential vorticity is always positive, and thus condition (26*a*), $(\alpha - U)\,\partial_y Q \geqslant 0$, with $\alpha \geqslant \max U \equiv U(0)$, is satisfied for any amplitude of the current. Thus, for the flow to be unstable, condition (26*b*) must be violated, i.e. it must be $(\alpha - U)^2 > g_r H$ in some region (which turns out to be in the neighbourhood of infinity), for which it follows that $U(0) > c$. (At the critical amplitude, it is $U(0) = c$ but $U(y) < g_r H(y))^{\frac{1}{2}}$, and both conditions (26) are satisfied for $\alpha = c$.) If the flow is unstable then the integral of $E_0 - \alpha P_0$ is not positive-definite for any value of α.

Finally, E_0 and P_0 conservation may assure inertial stability for amplitudes well beyond the maximum one for the Coriolis parameter and potential vorticity to be of equal sign. For instance, consider a very narrow westward jet with an exponentially decaying offshore structure, from a zonal coast where the Coriolis parameter is f_0. (Say, $f_0 > 0$ and $L \ll c/f_0, f_0/\beta$. The current may be in the equatorial or extra-equatorial region, depending on the value of $f_0^2/\beta c$.) For $U(0) < -f_0 L$ the Coriolis parameter and the potential vorticity have opposite signs in a region, next to the coast, of width $L \ln[-U(0)/Lf_0]$. However, the gradient of potential vorticity is positive for any (negative) amplitude $U(0)$, and total E_0 is positive-definite for $U(0) \geqslant -c + O(L^2)$. Therefore, if the flow is unstable, it must be $U(0) < -c + O(L^2) \ll -f_0 L$.

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Appendix. Evaluation of the stability boundaries

Consider the jet (35) with $f = \beta y$ and $f_0 = \beta y_0 \geqslant 0$ (the case $y_0 = 0$ corresponds to the Gaussian equatorial jet (34)). The function ϕ has values between 0, for $y^2 \to \infty$, and λ ($\lambda \geqslant -1$), at $y^2 = y_0^2$, and it satisfies

$$c\,\partial_y \phi = \mu f \phi \lambda, \quad c^2\,\partial_{yy}\phi = \mu(\mu f^2 - \beta c)\,\phi. \tag{A1}$$

The potential vorticity and its meridional gradient are given by

$$Q = \frac{f}{\bar{h}}\frac{1+\mu^2\phi}{1+\phi}, \tag{A2}$$

$$\partial_y Q = \frac{\beta\bar{h}}{H^2}[1 + \sigma(\mu, y)\,\phi + \mu^2\phi^2], \tag{A3}$$

where

$$\sigma(\mu, y) = 1 + \mu^2 + \frac{(1-\mu^2)\,\mu f^2}{\beta c}. \tag{A4}$$

Condition (*i*). In order to find the minimum amplitude for the expression between square brackets in (A 3) to vanish, we solve for $[\ldots] = 0$ and $\partial_y[\ldots] = 0$, which yields

$$\lambda\mu^2 = -1, \quad y^2 = 0, \tag{A 5}$$

valid for narrow easterlies ($\mu > 1$, $\lambda < 0$); and

$$\mu^2\phi = 1-\mu^2 \pm (1-\mu^2+\mu^4)^{\frac{1}{2}}, \quad (1-\mu^2)\mu\beta y^2/c = 1-\mu^2(3+2\phi) \tag{A 6}$$

(and λ calculated, from the values of ϕ and $\beta y^2/c$, using (35)), valid for wide easterlies ($\mu < 1$, $\lambda < 0$) and narrow westerlies ($\mu > 1$, $\lambda > 0$). If $y_0 \neq 0$ and $y^2 < y_0^2$ in (A 5) or (A 6), then the solution is that of $\partial_y Q(y_0) = 0$, i.e. $[\ldots] = 0$ in (A 3), with smallest $|\phi|$, viz

$$\lambda = -\frac{2/\sigma_0}{1+(1-(2\mu/\sigma_0)^2)^{\frac{1}{2}}}, \quad \sigma_0 = \sigma(\mu, y_0), \quad y^2 = y_0^2. \tag{A 7}$$

These equations must be used instead of (A 6) for wide easterlies ($\mu_0 \leqslant \mu < 1$, $\lambda < 0$) and narrow westerlies ($\mu \geqslant \mu_0' > 1$, $\lambda > 0$), where μ_0 and μ_0' are the two values of μ for which $y^2 = y_0^2$ in (A 6). For narrow easterlies (A 7) must be used instead of (A 5), and is valid as long as $2\mu > \sigma_0$, which happens for

$$(2\mu-\xi+1)^2 \leqslant 1-6\xi+\xi^2, \quad \xi = \frac{c}{\beta y^2} > 3+2\surd 2. \tag{A 8}$$

If $\beta y_0^2 > 3-2\surd 2$, then there are no narrow easterlies for which the gradient of potential vorticity changes sign.

Condition (*ii*). For the jet (35), conditions (27*b*) and (29*b*) are equivalent because the extrema of $U(y)$ and $H(y)$ occur at the same latitudes. Those conditions are satisfied for

$$\frac{-2}{1+(1+4\mu^2)^{\frac{1}{2}}} \leqslant \lambda \leqslant \frac{1}{\mu}. \tag{A 9}$$

Condition (*iii*). This is satisfied if $1+\mu^2\phi \geqslant 0$ in (A 2), which happens for

$$\lambda \geqslant \mu^{-2}, \quad \mu > 1 \quad (y^2 = y_0^2); \quad \text{all} \quad \lambda, \mu < 1. \tag{A 10}$$

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